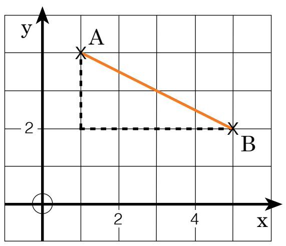
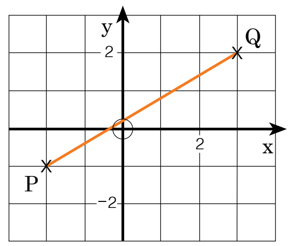
6 Line Segments

A Distance Between Two Points

1



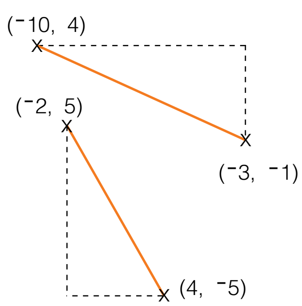


Use Pythagoras' Theorem, $a^2 + b^2 = c^2$, to calculate the length of the line segments joining . . .

a) $A\,(1,\ 4)$ and $B\,(5,\ 2)$

b) $P\,(-2,\ -1)$ and $Q\,(3,\ 2)$

2 Calculate the distance between . . .



a) $(-10,\ 4)$ and $(-3,\ -1)$

b) $(-2,\ 5)$ and $(4,\ -5)$

The distance between $A\,(x_1,\ y_1)$ and $B\,(x_2,\ y_2)$ is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

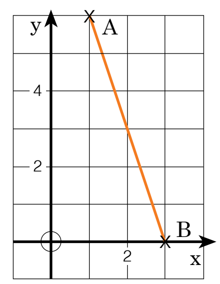
Note : the order of the coordinates does not matter, the answer will be the same.

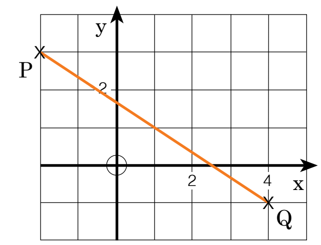
3 $P = (-6,\ 14)$, $Q = (5,\ -2)$
Calculate the length of the line segment PQ.

4 The distance between $(-3,\ 10)$ and $(5,\ k)$ is 17.
Calculate the possible values of k.

B Midpoint of a Line Segment

1





Read off the coordinates of the midpoint of the line segment joining.

a) $A\,(1,\ 6)$ and $B\,(3,\ 0)$

b) $P\,(-2,\ 3)$ and $Q\,(4,\ -1)$

The coordinates of the midpoint of a line segment joining $A\,(x_1,\ y_1)$ and $B\,(x_2,\ y_2)$ are $\left(\dfrac{x_1 + x_2}{2},\ \dfrac{y_1 + y_2}{2}\right)$.

2 Find the midpoint of the line segment joining . . .

a) $(-1,\ 3)$ and $(5,\ 7)$

b) $(-4,\ -10)$ and $(6,\ -15)$

3 AB is the diameter of a circle. $A = (-3,\ 6)$, $B = (9,\ 4)$

a) Find the coordinates of the centre of this circle.

b) Calculate the length of the radius.

4 $M\,(2,\ -3)$ is the midpoint of line PQ with $P = (-1,\ 2)$.

What are the coordinates of Q?

5 In ΔABC, $A = (1,\ -1)$, $B = (3,\ 3)$, $C = (6,\ -2)$.
Find the length of the median through C inside the triangle.

A Describe Features

Of course you can use your graphing calculator to draw trig graphs. Basic knowledge or what to expect is helpful when reading off features. Here are some things to remember when drawing trig graphs.

- Enter the GRAPH section, get into the SETUP menu, scroll to Angle and make sure it is set to degrees (Deg).
- Adjust the view-window. For instance, the x-values could go from $^-180$ to 360, scale 30.
  The y-values could go from $^-5$ to 5, scale 1, but that depends on the equation.
- When describing features, you could use your knowledge about the graphs or press F5 G-Solv and use ROOT , MAX or MIN .

Example : Describe features of the graph of $y = 0.5 + \sin(x - 60^\circ)$.

Method : Set the view-window as described above. Enter the equation and graph it.

Features : The range is $^-0.5 \le y \le 1.5$, the amplitude is 1, the period is 360˚, x-intercepts at . . . $^-90^\circ$, 30°, 270°, . . .
Maxima at 150˚ (±360˚), minima at 330˚ (±360˚).

1 In the table below describe features of the graphs A to H.

A $y = \cos(3x)$ B $y = 3 - \sin(2x)$ C $y = \cos(x + 30^\circ) + 1$ D $y = 20\sin\left(\frac{x}{2}\right)$

E $y = 2\cos x - 3$ F $y = -\sin(180x)$ G $y = 3\sin(x + 60^\circ)$ H $y = \cos\left(\frac{2x}{3}\right) + 2$

graph	range	amplitude	period	x-intercepts	maxima at	minima at
A						
B						
C						
D						
E						
F						
G						
H						

2 Draw the graphs of equations B and G above.

B

G

46 Sequences

A Next!

1 Write down the next two terms in these number sequences.

a) 2, 4, 6, 8, ..., ... b) 1, 2, 4, 8, ..., ...

c) 81, 27, 9, 3, ..., ... d) 1, 4, 9, 16, ..., ...

e) $^-8$, $^-2$, 4, 10, ..., ... f) 2, 3, 5, 8, ..., ...

g) 1, $^-1$, 1, $^-1$, ..., ... h) 0, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, ..., ...

A <u>sequence</u> is an ordered list of numbers, the first number is term 1, the next term 2, the nth term is called the <u>general term</u>. Notations for the general term are t_n or $T(n)$ or A_n etc.

In an <u>arithmetic sequence</u> (also called arithmetic progression or AP) each term after the first term is found by *adding* a constant amount to the previous term.

e.g. : $^-1$, 4, 9, 14, 19, . . . (add 5)

e.g. : 43, 40.5, 38, 35.5, . . . (add $^-2.5$)

In a <u>geometric sequence</u> (also called geometric progression or GP) each term after the first term is found by *multiplying* the previous term by a constant factor.

e.g. : 2, 8, 32, 128, 512, . . . (multiply by 4)

e.g. : 405, 135, 45, 15, 5, . . . (multiply by $\frac{1}{3}$)

2a) Which of the sequences in question 1 are APs?

....

b) Which are GPs ? ....

3 Explain why the sequence 4, 4, 4, 4, . . . is an AP as well as a GP.

....

....

....

4a) Write down the value of t_3 in these sequences.

i) Each term is half of the previous term and $t_1 = 100$

....

ii) Each term is 7 more than the previous term and $t_1 = {}^-18$

....

b) Explain why these rules are not very user-friendly if we want to find t_{100}.

....

....

....

B Direct Access

Usually the general term t_n is expressed in terms of n, which gives direct access to any specific term.

Examples :

a) $t_n = 2n + 6$, calculate t_1 and t_{50}

b) $T(n) = 3^n$, calculate $T(1)$ and $T(10)$

c) Calculate the 20th term of the sequence $<n^2 - 3n>$

Working :

a) $t_1 = 2 \times 1 + 6 = 8$; $t_{50} = 2 \times 50 + 6 = 106$

b) $T(1) = 3^1 = 3$; $T(10) = 3^{10} = 59049$

c) $t_{20} = 20^2 - 3 \times 20 = 340$

1 Write down the first 3 terms and the 20th term of these sequences.

a) $t_n = 4n - 15$ ...., ..., ... ; ....

b) $T(n) = 10^n$ ...., ..., ... ; ....

c) $< 3(1 - 2n) >$ ...., ..., ... ; ....

d) $A(n) = \frac{40}{2^n}$ ...., ..., ... ; ....

e) $T(n) = n^2 + n$ ...., ..., ... ; ....

f) $t_n = \frac{1}{n}$ ...., ..., ... ; ....

g) $A_n = \frac{24}{n^3}$ ...., ..., ... ; ....

h) $< 2(3^n) >$ ...., ..., ... ; ....

2a) Which of the sequences in question 1 are APs? Write down their formulas.

....

....

b) Which are GPs ? Write down their formulas.

....

....

3

linear	quadratic	rational	exponential

Complete the sentences with a word chosen from the box.

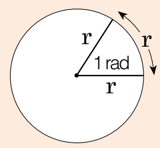
a) The general term of an AP is a .... function of n.

b) The general term of an GP is a .... function of n.

Radians

A Another Unit for Angle Size

An angle is the amount of space between two meeting lines. We have been using degrees to measure this space, we chose 90° as the amount of space in a right angle. We will now look at another method of measuring angles, it is called radian measure.

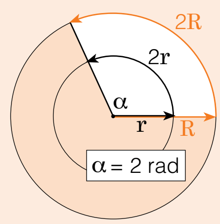


Definition : One radian (1 rad) is the angle at the centre of a circle subtending an arc whose length is equal to the radius.

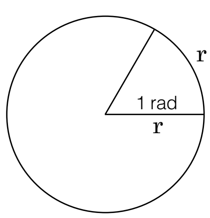
Notes :

- The size of the angle is independent of the choice of radius.
- The angle size is a number counting the number of radiuses that can be wrapped on the arc.

  angle $= \dfrac{\text{arc length}}{\text{radius}}$, so radians are just numbers without a unit.



1 Question : *"How many radians are there in one full turn?"*
The question can be rephrased as : *"How many radiuses can be fitted around the circumference of a circle?"*
Show that the answer equals 2π.



....

....

....

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2 Complete this table in which we convert degrees into radians, first as an exact number, then as a decimal approximation to 2 dp.

degrees	radians	
	exact	approximated
360°		6.28
180°	π	3.14
90°		
45°		
30°		
60°		
120°		
240°		
330°		

B Conversions

1a) If π rad = 180 degrees, how many degrees in 1 rad?

....

b) How many degrees in α radians?

....

2a) If 180 degrees = π rad, how many radians in 1 degree?

....

b) How many radians in β degrees?

....

Conversion formulas :

α rad $= \alpha \times \frac{180}{\pi}$ degrees; β degrees $= \beta \times \frac{\pi}{180}$ radians.

3 Convert these angles to radians.

a) 69° ....

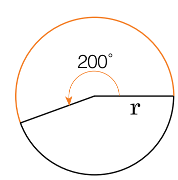
b) 315° ....

4 Convert these angles to degrees.

a) 2.5 rad ....

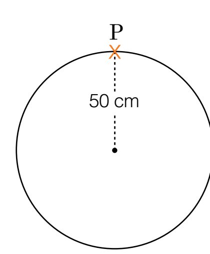
b) 8 rad ....

5 How many radiuses would fit on the red arc?



....

6 a) This wheel has a radius of 50 cm. The wheel is turned over 3.5 radians. What is the distance travelled by point P?



....

....

b) What if the wheel was turned 100 degrees?

....

....

7 Set your calculator to the radian mode and calculate these.

a) $\cos \frac{\pi}{3}$ .... b) $\tan \frac{5\pi}{4}$ ....

c) sin 1.2 .... d) cos 0.8 ....

Factorising 1

A From Sum to Product

Factorising is the reverse of expanding. When asked to factorise an expression you must write it as a product involving brackets. For example, the expression $2x + 4$ is the sum of two terms, $2x$ and 4. The expression $2x + 4$ can be written as $2(x + 2)$, which is the product of two factors, 2 and $x + 2$.

Examples : Factorise.

a) $5a^2 + 10a$ b) $a^2b + ab^2 - ab$

c) $x(y + 1) - x^2$ d) $a(b - 3) + c(b - 3)$

Working : Write the greatest common factor in front of the brackets, left overs go inside the brackets.

a) $5a^2 + 10a = 5a(a + 2)$

b) $a^2b + ab^2 - ab = ab(a + b - 1)$

c) $x(y + 1) - x^2 = x((y + 1) - x)$
$= x(y - x + 1)$

d) $a(b - 3) + c(b - 3) = (b - 3)(a + c)$

1 Factorise.

a) $6x - 9$ ....

b) $y^2 - xy$ ....

c) $3a^2 - 15a$ ....

d) $4ab - 8a$ ....

e) $a^2b^3 + a^3b^4 - a^2b^4$ ....

f) $6x^3 - 8x^2y$ ....

2 Factorise.

a) $8(a - 5) + 16$ ....

....

b) $a(b - 1) + 3a$ ....

....

c) $x(p - q) + 2(p - q)$ ....

....

....

d) $(a + b)^2 + 2(a + b)$ ....

....

e) $(x + y)^2 - x - y$ ....

....

....

B Two Stages

Example : Factorise $pr + qr - ps - qs$.

Working : There is no common factor in all four terms, but . . .

$pr + qr - ps - qs = r(p + q) - s(p + q)$
$= (p + q)(r - s)$ common factor

Some rearranging may be needed and remember :

$y + x = x + y$, $\quad {}^{-}x - y = {}^{-}(x + y)$, $\quad y - x = {}^{-}(x - y)$

1 Factorise.

a) $wy + wz + xy + xz$ ....

....

b) $ad - ae + dc - ec$ ....

....

c) $pr - qr - qs + ps$ ....

....

d) $2ab - bc + 2ad - cd$ ....

....

2 Factorise.

a) $ac + 2bd + ad + 2bc$

....

....

b) $2a^2 + 6ac - 6bc - 2ab$

....

....

c) $4wx - 3yz - 12xy + wz$

....

....

d) $10pq - 15pr + 6qr - 4q^2$

....

....

e) $ab + b - a - 1$

....

....

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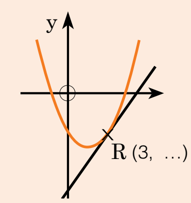
A Equation of the Tangent

The tangent to the curve is a sloped straight line. Its equation can be written in the form $y = mx + c$, where m is the gradient. The constant c can be calculated if we know a point on the line.

Example : A curve has equation $f(x) = x^2 - 4x - 5$. R is on the curve, its x-coordinate is 3. Find the equation of the tangent at R.



Working :

[Note : $f(3)$ will give us the y-coordinate of R;
$f'(3)$ will give us the gradient of the tangent at R.]

$f(3) = 3^2 - 4 \times 3 - 5 = {}^-8 \Rightarrow R = (3, {}^-8)$

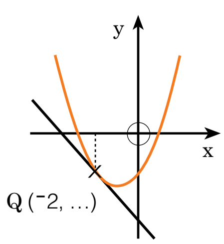
$f'(x) = 2x - 4$; $f'(3) = 2 \times 3 - 4 = 2 \Rightarrow m = 2$

The equation of tangent has the form $y = 2x + c$

Since $(3, {}^-8)$ is on the tangent $\quad {}^-8 = 2 \times 3 + c \Rightarrow {}^-14 = c$

Therefore the equation tangent at R is $y = 2x - 14$

1



The curve has equation $f(x) = x^2 + x - 6$. Q is on the curve, its x-coordinate is ${}^-2$. Follow these steps to write an equation of the tangent at Q.

a) Find the y-coordinate of Q. ....

b) Work out the gradient at Q. ....

....

c) Substitute values into the equation $y = mx + c$ and calculate c.

....

....

d) The equation of the tangent is ....

2 Given the polynomial with equation
$y = 0.5x^4 - x^3 - 4x^2 + 3x + 6$

a) Work out the coordinates of the y-intercept.

....

b) Find the gradient of the tangent at the y-intercept.

....

....

c) Write the equation of this tangent.

....

....

B PST . . .

1 A cubic has equation $f(x) = x^3 - 4x = x(x - 2)(x + 2)$.

a) Write an equation for the tangent at P $(1, {}^-3)$.

....

....

....

....

b) S is the smallest x-intercept of the cubic. Write an equation for the tangent at S.

....

....

....

....

....

2 T is on the parabola with equation $y = {}^-x^2 + 5x - 2$, its x-coordinate is 4. Find the equation of the tangent to the parabola at T.

....

....

....

....

....

3 There are two points Q and R on the cubic $f(x) = x^3 - 5x^2 + 5x + 4$, for which the gradient equals 2. Write the equation of the tangent at each of these points.

....

....

....

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A Study the Samples

Oliver attends a boys' college where large proportions of students are of Polynesian or Asian descent. Oliver investigates the question *"I wonder if at our school, at the age of 16, boys of one ethnic group tend to be taller than boys of another ethnic group."* Oliver takes three random samples of 16 year-old boys : 30 boys with Asian ethnicity, 30 boys of Polynesian descent and 40 boys with a European background.

key : 15 | 8 = 158 cm

Asian Boys

15	8
16	
16	5 5 6 6 6 7 7 8 9 9 9
17	0 0 1 1 1 3
17	5 6 6 7 8 8 8 9
18	0 0 2 2

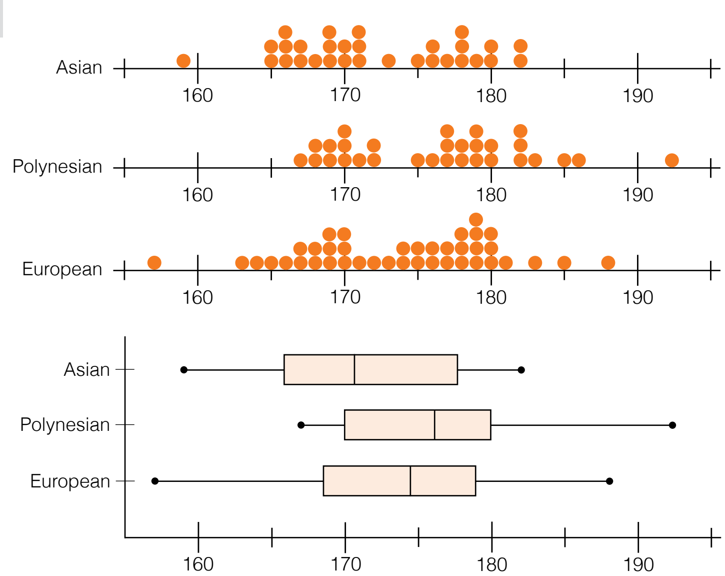
Polynesian Boys

16	7 8 8 9 9
17	0 0 0 1 2 2
17	5 6 7 7 7 8 8 9 9 9
18	0 0 2 2 2 3
18	5 6
19	2

European Boys

15	7
16	3 4
16	5 6 7 7 8 8 9 9 9
17	0 0 0 1 2 3 4 4
17	5 5 6 6 7 7 8 8 8 9 9 9 9
18	0 0 0 1 3
18	5 8

Boys (16 yrs) Heights in Centimetres



1 Study the graphs for the samples. Compare features of the sample distributions (shape, spread, middle 50%, shift, overlap, unusual or interesting features). Don't make any inferences.

170 Absolute Risk and Relative Risk 1

A Reading Two Way Tables

1 This table is based on the NZ General Social Survey (2008). It shows the financial well-being of a random sample of 1000 residents based on household income. Did householders think their income met their every day needs?

household income	not enough	enough	more than enough	total
Under \$30 000	37	100	6	143
\$30 000 - \$69 999	54	244	24	322
\$70 000 - \$99 999	27	143	26	196
\$100 000 or more	26	232	81	339
total	144	719	137	1000

Statistics New Zealand

a) Work out the percentage of the respondents who . . .

i) thought their income was more than enough. ....

ii) had a household income of \$100 000 plus. ....

iii) had a household income of \$100 000 or more and thought it was more than enough. ....

b) Based on this survey, find the probability that in 2008 a randomly chosen New Zealand resident . . .

i) had a household income of under \$30 000 and thought that their household income was not enough to meet their everyday needs.

....

ii) rated their household income as enough.

....

iii) had a household income of \$30 000 to \$69 999.

....

c) i) Find the probability that a resident with a household income under \$30 000 thought (s)he had more than enough.

....

ii) Find the probability that a resident who thought (s)he had not enough, had an income of \$30 000 to \$69 999.

....

iii) Calculate the probability that a resident thought the income was more than enough, given that (s)he had a household income of \$100 000 or more.

....

B Risk

Absolute risk (or just risk) is simply the probability of an event. Often risk is expressed as a rate per 100 or 1000.
Suppose the probability that a person under the age of 65 will catch the flu this winter is 0.05, then we can say, *'the risk for a person under the age of 65 getting the flu this winter is 0.05'* or *'the risk of a person under 65 getting the flu is 5 in 100.'*

Relative risk is a comparison between two different risk levels. Suppose we also know that the probability of a person aged 65+ getting the flu this winter is 0.09.
The relative risk of a person aged 65+ getting the flu compared to those under the age of 65 is $\frac{0.09}{0.05} = 1.8$.

There are two possible statements of interpretation :
'People aged 65 or over are 1.8 times as likely to get the flu this winter than those aged under 65.' or
'There is an 80% increase in the chances of getting the flu this winter for people aged 65+ compared to those under the age of 65.'

1 Last year at Bay College a total of 16 students ended up in hospital with injuries sustained inside or outside the school.

gender \ result	injured	not-injured	total
male	10	574	584
female	6	615	621
total	16	1189	1205

a) What proportion of students at Bay College were not injured?

....

b) What proportion of the injured students are girls?

....

c) i) What is the risk for a male student to get injured?

....

ii) Write the risk as a rate per 1000. ....

d) i) What is the risk for a female student to get injured?

....

ii) Write the risk as a rate per 1000. ....

e) i) Calculate the relative risk of a male student to get injured compared to a female student. (This means you write the female risk in the denominator.)

....

ii) Interpret the result. ....

....

....

....

A Satisfying Two Inequations

Example :

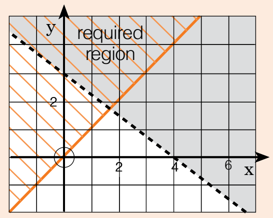
Clearly show the region where both inequations are valid.

$$\begin{cases} 3x + 4y > 12 \\ y \geq x \end{cases}$$

Working :

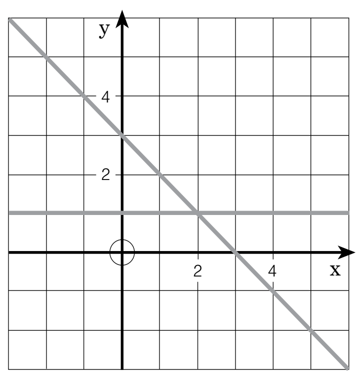
Draw a dotted line through (4, 0) and (0, 3). Shade above the line.
Draw a solid line through (0, 0) (2, 2). Shade above the line.

Now the required region is shaded **both** ways.

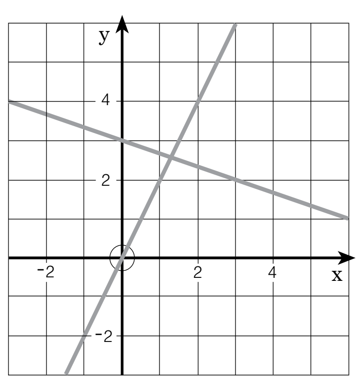


1 Shade the region which satisfies both inequations.

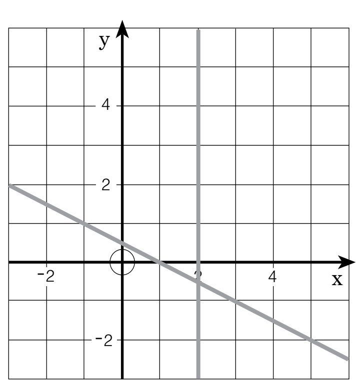
a) $x + y \leq 3$
$y > 1$



b) $y \leq \dfrac{9 - x}{3}$
$y < 2x$



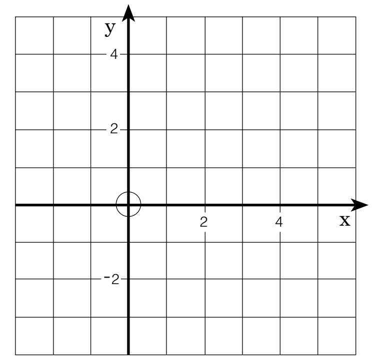
c) $x + 2y \geq 1$
$x < 2$



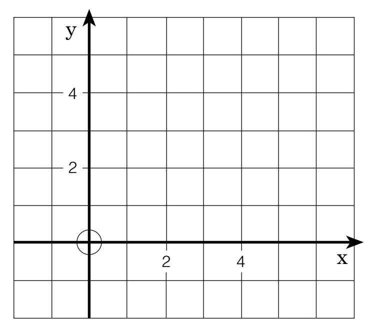
B More Drill

1 Clearly indicate the region which satisfies both inequations.

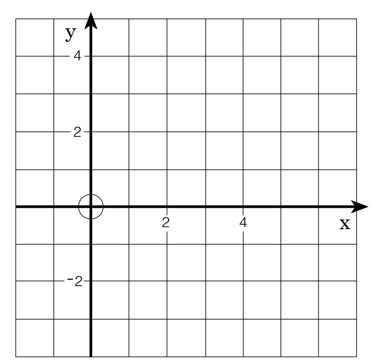
a) $2x - y \leq 3$
$y \leq 2$



b) $y \leq x + 2$
$5x + 3y \geq 12$



c) $x - 2y < 6$
$3x + 4y > 8$



2 $30a + 45b \geq 9000$
$3b \leq 2a + 150$

a and b are both positive numbers. Show the region defined by these constraints.