8 Manipulating Quadratics

A Completing the Square

The quadratic expression $x^2 + 6x + 9$ is a perfect square because it can be written as a square $(x + 3)^2$.

1 The expressions below are all perfect squares. Fill in the missing parts.

a) $x^2 -$ .... $+ 25 = (x - 5)^2$

b) $x^2 + 12x +$ .... $= (x + 6)^2$

c) $x^2 - 20x +$ .... $= (x -$ ....$)^2$

d) $x^2 +$ .... $+ 49 = (x +$ ....$)^2$

2 Complete the perfect squares.

a) $a^2 + 8a$ .... = (.......................)^2

b) $b^2 - 22b$ .... = (.......................)^2

c) $c^2 + 40c$ .... = (.......................)^2

d) $d^2 + 5d$ .... = (.......................)^2

e) $e^2 - e$ .... = (.......................)^2

f) $f^2 - 7f$ .... = (.......................)^2

g) $g^2 + 100g$ .... = (.......................)^2

Example : Write $x^2 + 8x + 5$ in the form $(x + b)^2 + c$.

Working : Concentrate on the first two terms of the quadratic and complete the square.

Since $x^2 + 8x + 16 = (x + 4)^2$

then $x^2 + 8x + 5 = (x^2 + 8x + 16) - 16 + 5$

$= (x + 4)^2 - 11$

3 Write these quadratics in the form $(x + b)^2 + c$.

a) $x^2 + 16x + 100 = (x^2 + 16x + 64)$ ....

$= (x + 8)^2 +$ ....

b) $x^2 + 6x + 10$ ....

....

c) $x^2 - 4x + 1$ ....

....

d) $x^2 - 10x - 5$ ....

....

B In the Form $a(x - b)^2 + c$

Example : Write $2x^2 + 12x - 5$ in the form $a(x - b)^2 + c$.

Working : $2x^2 + 12x - 5 = 2(x^2 + 6x) - 5$

$= 2(x^2 + 6x + 9) - 18 - 5$ (We add 2 lots of 9 therefore subtract 18)

$= 2(x + 3)^2 - 23$

1 Write these in the form $a(x - b)^2 + c$.

a) $3x^2 + 12x - 5$ ....

....

....

b) $2x^2 - 10x + 1$ ....

....

....

c) $^-4x^2 + 8x + 3$ ....

....

....

2 Sketch the graph of the function $y = 2x^2 + 6x + 1$.

....

....

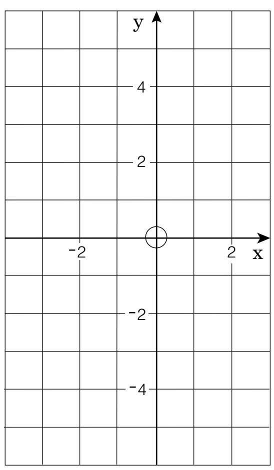
....

....

....

....

....



3a) $f(x) = 3x^2 - 3x - 8$. Write $f(x)$ in the form $a(x - b)^2 + c$.

....

....

b) i) For what value of x does $f(x)$ have its lowest value?
ii) What is this lowest value?

....

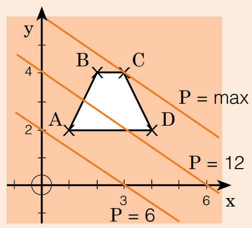
....

A Objective Functions Revisted

It is not necessary to work out the value of the objective function at each vertex. The gradient of the objective function can give us a clue for finding the optimum point.

Example :

ABCD is the feasible region. At which vertex does the objective function $P = 2x + 3y$ reach a maximum?



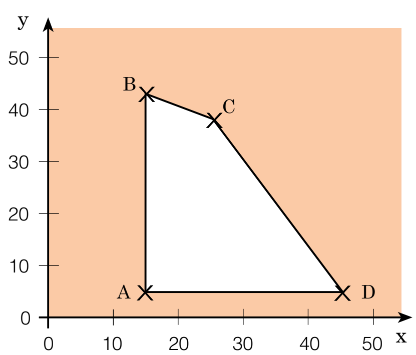
Working :

Choose convenient values for P and draw a few lines.

For instance . . .

$P = 6$, draw $2x + 3y = 6$
$P = 12$, draw $2x + 3y = 12$
The lines are parallel and the value of P increases as we move up. Therefore the maximum value of P will be found at vertex C.

1 ABCD is the feasible region, the exact coordinates of the vertices are unknown. At which vertex does the objective function $P = 3x + y$ reach a maximum?



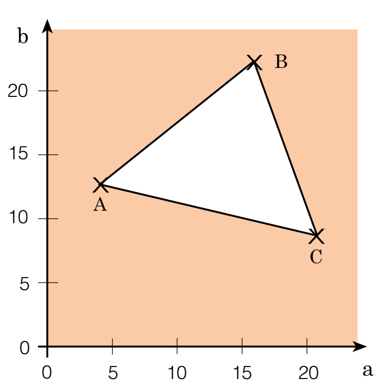
....

....

....

....

2



$Q = 3a + 5b$

The feasible region is left white, the coordinates of the vertices are unknown. At which vertex does the objective function Q reach a minimum?

....

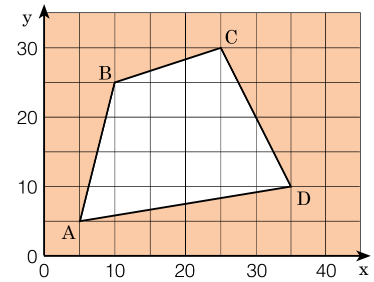
....

....

....

B Multiple Solutions

Region ABCD is the feasible region for the three questions below.



1 The objective function is $P = 2x + y$.

a) Which *two* vertices maximise P?

....

....

....

b) What is the maximum value of P? ....

....

c) Show that at point (30, 20) P has the same maximum value.

....

d) If x and y must be whole numbers, how many points (x, y) maximise P?

....

2 When optimising $Q = 4x - y$ over the region above, we can also find multiple solutions.

a) Describe where these solutions will be found.

....

....

b) Do these points maximise Q or do they minimise Q?

....

....

....

3 Make up an objective function R which will have optimum values on line segment BC. Are these maxima or minima?

....

....

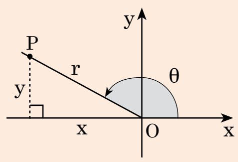
....

....

....

48 Trig Ratios 1

A Sine, Cosine and Tangent



Definitions of sin θ, cos θ, tan θ

Consider point P(x, y) in the Cartesian plane. Distance OP = r and the angle between OP and the positive x-axis is θ. (θ is measured anti-clockwise and can be any size)

We define these ratios : $\sin\theta = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$; $\tan\theta = \frac{y}{x}$.

Note : x and y are coordinates which can be positive or negative depending on the size of θ;
r, a distance, is always positive.

Examples : a) Without using a calculator, predict whether $\cos 295^\circ$ is positive or negative.

b) Given A = (−2, −3) ; the angle between the positive x-axis and OA is θ. Write sin θ as a decimal to 4 dp.

Working : a) When θ = 295°, P lies in the 4th quadrant i.e. x is positive, y is negative. Therefore

$\cos 295^\circ = \frac{x}{r} = \frac{\text{positive}}{\text{positive}} = \text{positive}.$

b) With A = (−2, −3), y = −3 and

$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}.$

Hence $\sin\theta = \frac{y}{r} = \frac{^-3}{\sqrt{13}} = {}^-0.8321$ (4 dp)

1 Predict whether these ratios are positive or negative.

a) sin 200° .... b) tan 15° ....

c) cos ⁻40° .... d) sin ⁻120° ....

e) tan 500° .... f) cos 750° ....

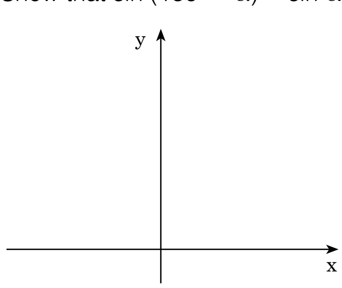
2 P = (3, ⁻4) and θ is the angle between OP and the positive x-axis. Calculate sin θ, cos θ and tan θ.

....

....

....

3 Show that sin (180° − α) = sin α.



....

....

....

....

....

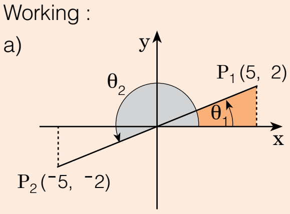
B Going Around in Circles

Examples : a) Given that $\tan\theta = \frac{2}{5}$, draw 2 possible angles θ in the Cartesian plane.

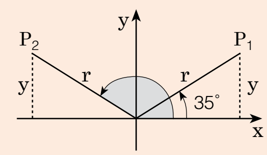
b) If sin α = sin 35°, find 4 possible sizes of angle α.

Working :

a)

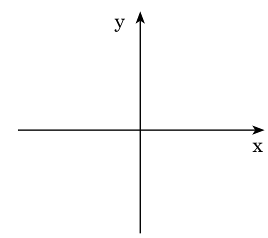


b)



α can be 35°
or 180° − 35° = 145°
or 35° + 360° = 395°
or 145° + 360° = 505°

1 Given $\cos\theta = \frac{^-3}{5}$.



a) Draw 2 possible angles θ in the Cartesian plane.

b) If also given that θ is an obtuse angle, calculate sin θ and tan θ.

....

....

2 Given reflex angle β with $\tan\beta = \frac{5}{12}$. Find sin β and cos β.

....

....

3 Find 4 possible sizes of angles α, β and γ.

a) tan α = tan 25°

....

....

b) sin β = sin 210°

....

....

c) cos γ = cos ⁻75°

....

....

Fast Track 5M Mathematics Workbook - 1st Edition © Sigma Publications Ltd 2012 ISBN 978-1-877567-15-5. A Copyright Licensing Ltd licence is required to copy any part of this resource.

A Delaying a Task

The *float of a task* is the amount of time the task can be delayed or extended without causing delay to the finishing time of the entire project i.e. Float of a task is the difference between early start and late start of the task. Float = LS - ES

Note : the word *slack* is sometimes used instead or float.

1 Complete this activity table, showing starting and finishing times of Project 1 on the previous page. The order of the tasks is taken from the columns in the network.

Task ID	Time span	Starting ES	Starting LS	Finishing EF	Finishing LF	Float	Critical? (Y / N)
A	3	0	2	3	5	2	N
B	4	....	....	....	....	....	....
E	6	....	....	....	....	....	....
C	5	....	....	....	....	....	....
D	6	....	....	....	....	....	....
F	4	....	....	....	....	....	....
G	5	....	....	....	....	....	....
H	3	....	....	....	....	....	....
J	1	....	....	....	....	....	....

2 Read each statement and decide whether it is true or false. (T or F)

a) There is no float time on a critical path. ....

b) There can be only one critical path in a project. ....

c) The earliest starting time of a task equals the minimum of the earliest finishing times of its predecessors. ....

d) The latest finishing time of a task equals the minimum of the latest starting times of the tasks following it. ....

e) The float equals the difference between earliest and latest finishing times. ....

f) The earliest times always overlap with the latest times i.e. EF cannot be lower than LS. ....

B Making a Time Chart

A time chart shows what activities are running simultaneously. It shows what tasks are active at any time and can be used to schedule resources such as personnel or equipment.

There are many ways to draw these time charts, examples can be found on the internet. Google *Gantt Chart*.

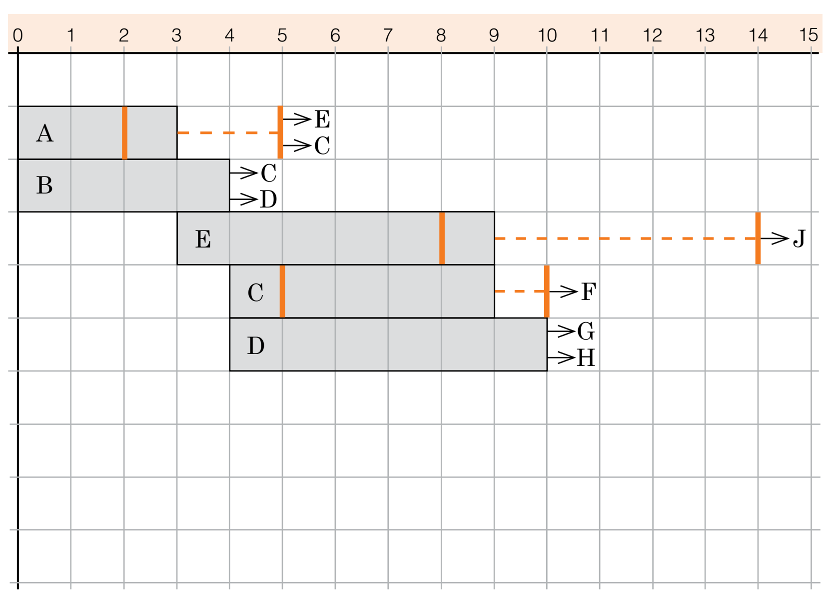
1 On the right is a time chart used for the planning of Project 1 (see A and page 82) The time line runs from 0 to 15 days.

a) How is the float shown?

....

b) Complete the time chart with activities F, G, H and J.

Time Chart Project 1 (time in days)



2 Each activity in this project is meant for one person and can be done by any of the 5 office workers in the team. What is the minimum number of employees needed to finish the project? Suggest a possible allocation of tasks.

....

....

....

....

94 Manipulating Surds 2

A Rational Denominators

When simplifying divisions containing surds, the convention is that there must be no surds left in the denominator. We can rationalise the denominator by multiplying both numerator and denominator with the same number, as a rule this number is either a surd or a conjugate surd.

Examples : Write these in simplest form with a rational denominator.

a) $\dfrac{3+\sqrt{2}}{2\sqrt{6}}$ b) $\dfrac{3+2\sqrt{3}}{5-\sqrt{3}}$

Working :

a) $\dfrac{(3+\sqrt{2})\times\sqrt{6}}{2\sqrt{6}\times\sqrt{6}} = \dfrac{3\sqrt{6}+\sqrt{12}}{12} = \dfrac{3\sqrt{6}+2\sqrt{3}}{12}$

b) $\dfrac{(3+2\sqrt{3})\times(5+\sqrt{3})}{(5-\sqrt{3})\times(5+\sqrt{3})} = \dfrac{15+3\sqrt{3}+10\sqrt{3}+6}{25-3}$

$= \dfrac{21+13\sqrt{3}}{22}$

1 Write these in simplest form with a rational denominator.

a) $\dfrac{2+\sqrt{5}}{\sqrt{2}}$ ....

b) $\dfrac{\sqrt{2}+\sqrt{3}}{2\sqrt{3}}$ ....

c) $\dfrac{4}{1-\sqrt{2}}$ ....

....

d) $\dfrac{\sqrt{2}}{3\sqrt{2}+1}$ ....

....

e) $\dfrac{3\sqrt{5}+2}{2\sqrt{5}-4}$ ....

....

....

2 Write $\dfrac{2-4\sqrt{3}}{5+3\sqrt{3}}$ in the form $a+b\sqrt{3}$, with $a, b \in I$.

....

....

....

B Using Multiplication Boxes

Use multiplication boxes when expanding brackets with three or more terms.

1 Expand and simplify $(a+1-3\sqrt{2})(a-2+4\sqrt{2})$.

	a	-2	$4\sqrt{2}$
a			
1			
$-3\sqrt{2}$			

....

2 Rationalise the denominator of these expressions.

a) $\dfrac{(a+1-\sqrt{b})}{(b-\sqrt{b})}$ ....

....

b) $\dfrac{a-1+\sqrt{a}}{a+1+\sqrt{a}}$ ....

....

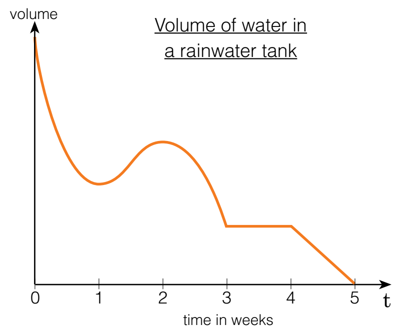
Fast Track 5M Mathematics Workbook - 1st Edition © Sigma Publications Ltd 2012 ISBN 978-1-877567-15-5. A Copyright Licensing Ltd licence is required to copy any part of this resource.

Gradient of a Curve

A Rate of Change

Calculus is the branch of mathematics concerned with rates of change. We will study how one variable changes compared with changes in the other variable.

1 This graph shows the volume of water in a rain water tank over a period of 5 weeks. By studying the steepness of the graph we can comment on how the volume of the water changes with time.



a) Match each description with a week.
(Note : $t = 0$ to $t = 1$ is week 1)

description	week
i) The volume of water remains unchanged.	....
ii) The volume of water increases.	....
iii) The volume of water is changing at a constant rate.	....
iv) The volume of water decreases fastest at the end of the week.	....

b) One of the five weeks has not featured in (a). Give a description of the change in volume during this week.

....

....

....

c) Complete these sentences.

zero	positive	negative

i) If the volume is increasing at time t, then the graph has a .... gradient.

ii) If the volume is decreasing at time t, then the graph has a .... gradient.

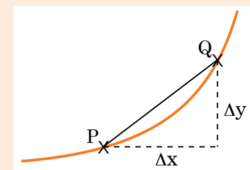
iii) If the volume is stationary at time t, then the graph has a .... gradient.

B Tangent

A gradient is a measure of steepness, it measures how fast y is changing in relation to x. We know the gradient of a line is

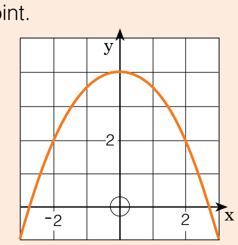
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

The gradient of a straight line is the same at any point.
The gradient of a curve keeps changing. For instance, in the diagram P and Q are on a curve very close together. The gradient of the curve between P and Q can be approximated by the straight line PQ with gradient $\frac{\Delta y}{\Delta x}$.
(Δ is the Greek letter d and is used to indicate small increments.)

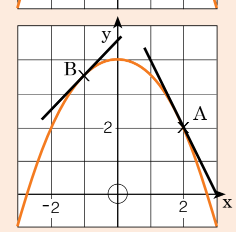


If we choose Q very very close to P, then the line PQ is almost the same as the tangent at P.
Results :
1. The notation for the gradient of a curve is $\frac{dy}{dx}$.
2 The gradient of a curve at a certain point equals the gradient of the tangent to the curve at that point.

Example :
This is the graph of $y = 4 - \frac{1}{2}x^2$.
a) Estimate the gradient of the curve at (2, 2).
b) Estimate the gradient when $x = {}^-1$.



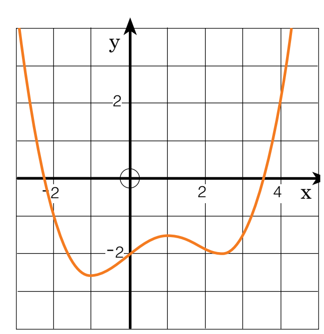
Working :
a) Draw the tangent at A(2, 2).
$\frac{dy}{dx} = \frac{-2}{1} = {}^-2$.
b) Go up from $x = {}^-1$ to meet the graph at B. Draw the tangent.
$\frac{dy}{dx} = \frac{1}{1} = 1$.



1 This is the graph of a polynomial of degree 4.

a) Estimate the gradient of the curve at (0, ⁻2). ....

b) Estimate the gradient when $x = 3.5$. ....



c) For what values of x is the gradient zero?

....

d) For what values of x is the curve increasing?

....

e) For what values of x is the curve decreasing?

....

A Marginal

If $Q(x)$ represents a sales quantity like cost, revenue profit or loss on the sale of x items, then $\frac{dQ}{dx}$ (the rate of change in Q) is called the marginal quantity.

For instance : $R(100)$ gives the revenue (in dollars) on the sale of the first 100 items, $R'(100)$ gives the marginal revenue (in dollars per item) of the 100th item - that is the increase in revenue when the 100th item is sold.

1 The marginal cost (in dollars) for the x^{th} item sold by a factory is given by $\frac{dC}{dx} = 0.4x^{-\frac{1}{2}} + 0.1$.

a) Find the marginal cost for . . .

i) the first item. ....

ii) the 100th item. ....

b) Given that the cost for the first 1000 items is 500 dollars, write an equation for the cost function $C(x)$.

....

....

....

....

....

2 The marginal profit for the sale of the first x computers given by $\frac{dP}{dx} = (1 + 0.2x)^2$. The profit for the first 100 computers is \$16 000. Find the profit when 200 computers are sold.

....

....

....

....

....

....

....

....

....

....

B On the Edge

1 The flow of a certain river is fastest in the middle and slowest at the river bank where friction slows the water down. If x is the distance from midstream, then the rate of change in velocity with respect to x is given by $\frac{dv}{dx} = {}^-3e^{-1.5x}$.

Midstream velocity is 2.8 ms^{-1}, the river is 6 m wide. How fast is the flow at the river banks?

....

....

....

....

....

2 The second derivative of a function is $f''(x) = 6x^2 - 6x + 4$. The graph of $y = f(x)$ has a y-intercept at 5 and a turning point for $x = 3$. Find the equation of $f(x)$.

....

....

....

....

....

....

....

3 When a balloon is filled with gas, the rate of change in volume is given by $\frac{dV}{dt} = \frac{8}{V}$ litres per second.

The initial volume of the balloon is 0 litres. Write an equation for the volume of the balloon after t seconds.

....

....

....

....

....

A Eliminate One by One

If a situation involves three different variables, x, y and z, then we need three simultaneous equations to find a unique solution.
We will use the elimination method as follows :

Step 1 take the first two equations and eliminate x, giving an equation in y and z.
Step 2 take the last two equations and eliminate x, giving another equation in y and z.
Step 3 two equations with two variables (y and z) can be solved by eliminating y, giving us a solution for z.
Step 4 substitute z into one of the equations in y and z and solve for y.
Step 5 substitute y and z into one of the original equations and solve for x.

Example :

Solve
$$x + 2y + 3z = 13 \quad ①$$
$$2x + 2y + z = 10 \quad ②$$
$$3x + y - 2z = 5 \quad ③$$

Working :

Step 1 - eliminate x from ① and ②
①x 2 $\quad 2x + 4y + 6z = 26$
② $\quad \ominus\ 2x + 2y + z = 10$
$\quad 2y + 5z = 16 \quad ④$

Step 2 - eliminate x from ② and ③
②x 3 $\quad 6x + 6y + 3z = 30$
③x 2 $\quad \ominus\ 6x + 2y - 4z = 10$
$\quad 4y + 7z = 20 \quad ⑤$

Step 3 - eliminate y from ④ and ⑤
④x 2 $\quad 4y + 10z = 32$
⑤ $\quad \ominus\ 4y + 7z = 20$
$\quad 3z = 12$
$\quad z = 4$

Step 4 - substitute z into ④
$2y + 5 \times 4 = 16$
$2y = {}^{-}4$
$y = {}^{-}2$

Step 5 - substitute y and z into ①
$x + 2 \times {}^{-}2 + 3 \times 4 = 13$
$x = 5$

Solution : $x = 5$, $y = {}^{-}2$, $z = 4$

Check your answer by substitution into ② and ③
② $2 \times 5 + 2 \times {}^{-}2 + 4 = 10$ ✓ ③ $3 \times 5 + {}^{-}2 - 2 \times 4 = 5$ ✓

1 Solve simultaneously.

a)
$$x + 2y + z = 7$$
$$2x + 3y + 3z = 8$$
$$4x - y - 2z = 4$$

b)
$$3x - 2y + z = 2$$
$${}^{-}3x + y - 2z = 2$$
$$2x + 3y + 3z = {}^{-}10$$

c)
$$2x + y + z = {}^{-}2$$
$$x + 3y - z = 7$$
$$3x - y - 4z = 3$$