138 Derivative at a Point

A Gradient at a Point on the Curve

Remember that the derived function $f'(x)$ is also called gradient function. For the curve $y = f(x)$ the gradient at a point with x-coordinate a, is equal to $f'(a)$.

Example : Find the gradient of the curve $f(x) = (2x - 1)^3$ at the point with x-coordinate 2.

Working : Find $f'(x)$ using the chain rule.
$f'(x) = 3(2x - 1)^2 \times 2 = 6(2x - 1)^2$.
At $x = 2$, the gradient equals $f'(2) = 6 \times 3^2 = 54$

1 For each of the following curves find the gradient at point P.

a) $f(x) = x^3 - 2x^2 - 5x + 4$; P has x-coordinate 3.

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b) $g(x) = 3x + \frac{2}{x}$; P = (2, 7).

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c) $h(x) = (x^2 - 5x + 1)^2$; P is the y-intercept of the curve.

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2 $V = \frac{4}{3}\pi r^3$. Find the rate of change in V with respect to r, when $r = 5$.

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3 $P(t) = \sqrt[3]{t^2 + 2t}$. Find $\frac{dP}{dt}$ when $t = 2$.

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B Just Checking

We can check the derivative at a point with the GC.

Example : In the example in column A we found that the gradient of the curve $f(x) = (2x - 1)^3$ at $x = 2$ equals 54. Check this with your GC.

Working : Select [TABLE] from the main menu; enter $Y_1 = (2x - 1)\wedge 3$; select [SET] and make sure the x-values include 2 (for instance Start : $^-2$, End : 5, Step : 1). Now select [TABL] to display the table showing X, Y_1 and Y'_1.

Check : For X = 2 we find $Y'_1 = 54$. Correct ✓.

1 Check your answers to column A, question 1. ✓ ✗

a) X = ...***3***....... Y_1 = ...***$^-$2***..... Y'_1 = ... ☐

b) X = ... Y_1 = ... Y'_1 = ... ☐

c) X = ... Y_1 = ... Y'_1 = ... ☐

2 Check your answers to column A, question 3.

X = ... Y_1 = ... Y'_1 = ... ☐

3a) Find the derivative of the function $y = \sqrt{2x - 4}$.

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b) Find the gradient at $x = 3$.

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c) Check the gradient at $x = 3$.

X = ... Y_1 = ... Y'_1 = ...

Correct? ....

d) Explain why Y_1 shows ERROR for X = 0 and X = 1.

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e) Explain why Y'_1 shows ERROR for X = 2.

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Not all curves are described with an explicit equation of the form $y = f(x)$. In AS 3.1 we explored a range of curves which are either written in implicit form or in parametric form. For instance, the equation of a circle with centre (0, 0) and radius 4 can be written in implicit form as $x^2 + y^2 = 16$ or in parametric form as $x = 4\cos t$, $y = 4\sin t$. On this page we will work out the gradient function $\frac{dy}{dx}$ of curves that are written in parametric form.

A Path

Parametric equations are often used to describe the path of an object. The coordinates (x, y) of the path are expressed as a function of time : $x = f(t)$, $y = g(t)$.

$\frac{dy}{dx}$ is found with the chain rule : $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

where $\frac{dy}{dt} = g'(t)$ and via $\frac{dx}{dt} = f'(t)$ we find $\frac{dt}{dx} = \frac{1}{f'(t)}$.

Example : A particle moves on a path given by $x = t^2$, $y = 5t$

a) Express the gradient, $\frac{dy}{dx}$, in terms of t.

b) Calculate the gradient at time $t = 4$.

Working : a) $y = 5t \Rightarrow \frac{dy}{dt} = 5$

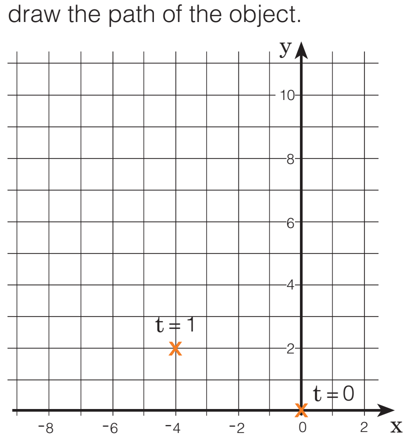
$x = t^2 \Rightarrow \frac{dx}{dt} = 2t \Rightarrow \frac{dt}{dx} = \frac{1}{2t}$

then, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 5 \times \frac{1}{2t} = \frac{5}{2t}$

b) At time $t = 4$, the gradient is $\frac{5}{2 \times 4} = \frac{5}{8}$

1 The path of an object is given as $x = t^2 - 5t$, $y = 2t$.

a) Complete the table, plot points and draw the path of the object.



t	x	y
0	0	0
1	-4	2
2		
3		
4		
5		

b) Express the gradient, $\frac{dy}{dx}$, in terms of t.

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c) What is the gradient at time $t = 4$?

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B Around the Tracks

1 Parametric equations are given for 3 curves. For each find the gradient function $\frac{dy}{dx}$, in terms of t.

a) $x = t^2 + 1$, $y = 3t - 1$ ....

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b) $x = \frac{1}{t}$, $y = \frac{t^3}{3}$ ....

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c) $x = \cos t$, $y = \sin 2t$ ....

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2 These are parametric equations for an ellipse.

$x = 3\cos t$, $y = 5\sin t$.

a) Point P is on the ellipse, the t-value of P is 4. Plot point P.

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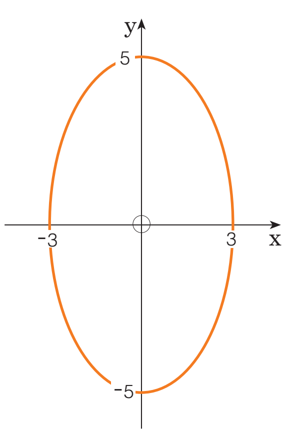

b) Calculate the gradient of the ellipse at P.

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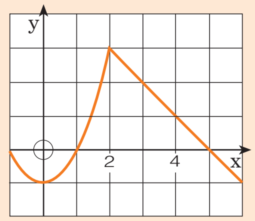
A Differentiability

A function is differentiable if the derivative can be found. When there is a hole or jump in the curve then the gradient can not be found at that point. Therefore points of discontinuity are also points of non-differentiability.

There are situations where the function is continuous but not differentiable at $x = a$. For instance this piecewise function is continuous, it has a spike at $x = 2$. Coming towards 2 from below, the parabola has a gradient of 4; coming towards 2 from above the line has gradient $^-1$.

Conclusion : The function is not differentiable for the spike at $x = 2$.



1 Piecewise function $f(x)$ is defined as follows :

$$f(x) = \begin{cases} 1 - x, & x \le 0 \\ x^2 - 4, & 0 < x < 3 \\ 2x - 1, & x \ge 3 \end{cases}$$

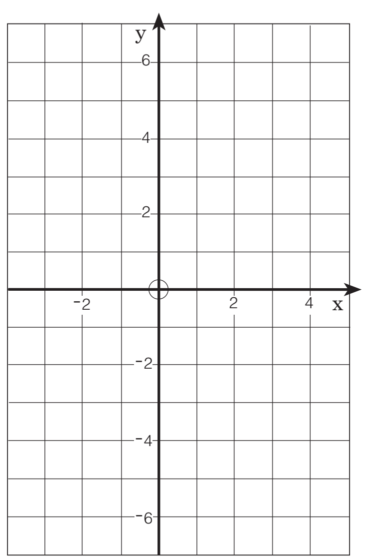
a) Complete

$f(^-2)$ = ...

$f(2)$ = ...

$f(4)$ = ...

b) Draw the graph of $y = f(x)$.



c) Complete

$f'(^-2)$ = ...

$f'(2)$ = ...

$f'(4)$ = ...

d) Find the values of x for which . . .

i) $f(x)$ is discontinuous ....

ii) $f(x)$ is not differentiable ....

2a) Which of these four functions has a point where $f(x)$ is continuous but not differentiable?

A $f(x) = \frac{1}{x}$ B $f(x) = |x|$

C $f(x) = e^x$ D $f(x) = \tan x$

b) Explain your choice. ....

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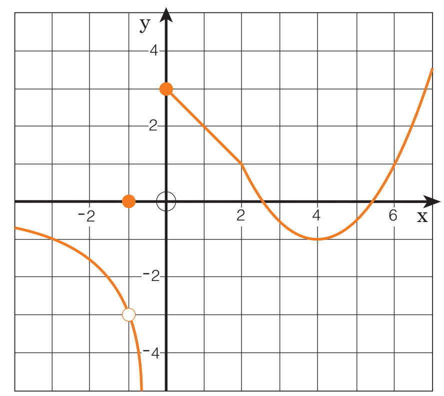
B Examining a Graph

1 This is the graph of piecewise function $y = f(x)$.



a) Read off the function values :

i) $f(^-3)$ .... ii) $f(^-1)$ ....

iii) $f(0)$ .... iv) $f(1)$ ....

v) $f(2)$ .... vi) $f(3)$ ....

vii) $f(4)$ .... viii) $f(6)$ ....

b) "The domain of the function f is $x \in R$."
Do you agree with this statement? Say why.

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c) Describe the range of f. ....

d) Find these limits (if they exist).

i) $\lim_{x \to ^-\infty} f(x)$ .... ii) $\lim_{x \to ^-1} f(x)$ ....

i) $\lim_{x \to 0} f(x)$ .... ii) $\lim_{x \to 2} f(x)$ ....

e) For what values of x is the function discontinuous?

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f) Use the graph to estimate these derivatives (where possible).

i) $f'(^-3)$ .... ii) $f'(^-1)$ ....

iii) $f'(0)$ .... iv) $f'(1)$ ....

v) $f'(2)$ .... vi) $f'(3)$ ....

vii) $f'(4)$ .... viii) $f'(6)$ ....

g) For what values of x is the function not differentiable?

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Using Substitution 1

The integral $\int f(x)\,dx$ is not always easily guessed. It may help to simplify $f(x)$ to $f(u)$ where u is a function of x. When changing $f(x)$ to $f(u)$ we must also manipulate dx to become du. Only when the whole integral is in terms of u can we take the integration step.

A Bracketed Polynomials

Examples : Find

a) $\int 4(2x-1)^8\,dx$ b) $\int x(x^2+4)^5\,dx$

Working :

a) Let $u = 2x - 1$,

then $\frac{du}{dx} = 2$

which gives $du = 2\,dx$

or $2\,du = 4\,dx$

Note : Although $\frac{du}{dx}$ is a notation meaning u', we can manipulate it like a fraction.

Now we replace $2x - 1$ by u and $4\,dx$ by $2\,du$.

$$\int 4(2x-1)^8\,dx = \int 2u^8\,du.$$
$$= 2 \times \tfrac{1}{9}u^9 + c = \tfrac{2}{9}(2x-1)^9 + c.$$

b) Let $u = x^2 + 4$, the integral still contains $x\,dx$.

Since $\frac{du}{dx} = 2x \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

Now substitute $x^2 + 4$ by u and $x\,dx$ by $\frac{1}{2}du$.

$$\int x(x^2+4)^5\,dx = \int \tfrac{1}{2}u^5\,du.$$
$$= \tfrac{1}{2} \times \tfrac{1}{6}u^6 + c = \tfrac{1}{12}(x^2+4)^6 + c.$$

1 Use substitution to find these integrals.

a) $\int (4x-2)^5\,dx$, let $u = 4x - 2$, then

$\frac{du}{dx}$ = ... $\Rightarrow$ dx = ... du.

Write the integral in terms of u and integrate :

$\int (4x-2)^5\,dx = \int$ ....

= .... (in terms of u)

= .... (in terms of x)

b) $\int x^2(x^3-1)^3\,dx$, let $u = x^3 - 1$, then

$\frac{du}{dx}$ = ... $\Rightarrow$ $x^2\,dx$ = ... du.

$\int x^2(x^3-1)^3\,dx$ = ...

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B Other Functions

Example :

Find the integral $\int e^x(e^x-2)^3\,dx$, by substituting $u = e^x - 2$.

Working : If $u = e^x - 2$, then $\frac{du}{dx} = e^x$ hence $e^x\,dx = du$

Now substitute into the integral giving . . .

$$\int u^3\,du = \tfrac{1}{4}u^4 + c = \tfrac{1}{4}(e^x-2)^4 + c.$$

1 Use substitution to find these integrals.

a) $\int 2x\cos(x^2-4)\,dx$, $u = x^2 - 4$

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b) $\int \frac{6x}{(x^2+1)^2}\,dx$, $u = x^2 + 1$

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c) $\int x^3 e^{x^4}\,dx$, $u = x^4$

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d) $\int \frac{2x-1}{x^2-x}\,dx$, $u = x^2 - x$

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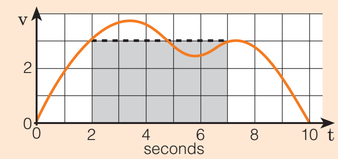
A Distance as an Area Under the Velocity Curve

The graph on the right shows the velocity of an object over 10 seconds.

Question : What distance did the object travel in time interval [2, 7] sec?

Answer : The average velocity in that timespan (shown by the dotted line) is 3 ms^{-1}, so in 5 seconds the distance travelled is about 15 m.



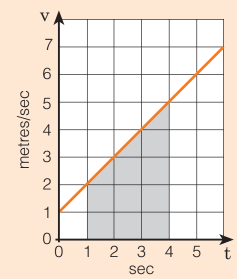
If the velocity of an object is positive between times $t = a$ and $t = b$ then the distance travelled can be found by calculating the area between the velocity curve and the horizontal t-axis, bounded vertically by the lines $t = a$ and $t = b$.

Example : An object moves on a track with velocity $v(t) = t + 1$ ms^{-1}

a) Work out an equation for displacement function $s(t)$.

b) Find the distance travelled in the 3 seconds from $t = 1$ to $t = 4$ by calculating $s(4) - s(1)$.

c) Use the grid to work out the area between the graphed line and the t-axis, bounded by the lines $t = 1$ to $t = 4$.



Working : a) $s(t) = \frac{1}{2}t^2 + t + c$.

b) $s(1) = \frac{3}{2} + c$ and $s(4) = 12 + c$; distance $= s(4) - s(1) = (12 + c) - (\frac{3}{2} + c) = 10\frac{1}{2}$ m.

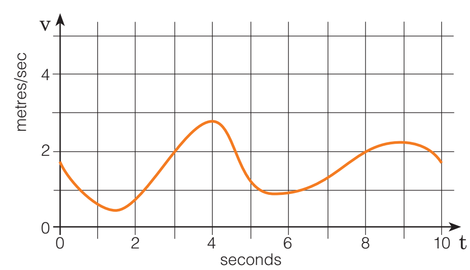
c) The required area is shaded grey. It is a trapezium with area $10\frac{1}{2}$ grid squares.

Note that the 'unit' for a grid square is sec x $\frac{\text{metres}}{\text{sec}}$ = metres, as required for displacement.

This example establishes a link between definite integral and area under a curve : If $v(t)$ is positive over $[a, b]$, then ...

$\int_a^b v(t)\, dt$ = distance travelled in timespan [a, b] = area bounded by the velocity curve, t-axis and vertical lines $t = a$ and $t = b$.

1 The graph shows the velocity of a slot-car as it does one lap on a track.



a) Describe the track of the slot-car. How many straights, how many corners? Give reasons for your conclusions.

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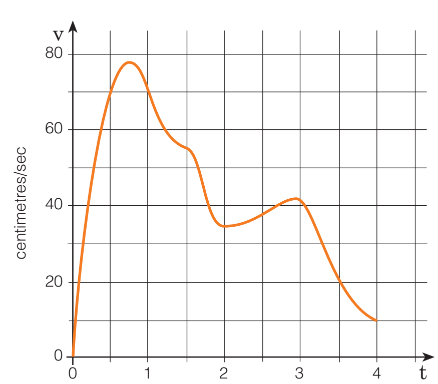
b) At $t = 2$ it starts the main straight and at $t = 5$ it reaches the end of the straight.

i) Shade the area that represents the displacement while on the main straight.

ii) Estimate the length of the straight. ....

c) Estimate the length of one lap. ....

2 The graph shows the velocity of a marble as it moves through obstacles on a track. After 4 seconds it falls into a hole at the end of the track.



a) What exactly does each grid square on this graph represent?

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b) What distance does the marble travel in the first second?

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c) Estimate the total length of the track. ....

A Manipulating Equations

1 Here are the equations of three different planes :

① $3x - 4y + z = {}^-11$
② ${}^-x + 5y - 2z = 18$
③ $2x + y - z = 7$

a) Are any of the planes parallel? How do you know?

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b) Points (1, 3, ⁻2) and (4, 8, 9) are solutions of the system of simultaneous equations. What does this tell you about the position of the three planes? Explain.

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Check that in the above system, equation ③ could be obtained by adding equations ① and ②.
In general, if in a set of 3 equations with 3 variables, one equation can be obtained by manipulating the other two, then we have a consistent, dependent system.

c) i) How does the solution set change if we change equation equation ③ into $2x + y - z = 12$?

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ii) Draw the set of planes and name the type of system.

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B Making It Happen

1 The solution to this set of simultaneous equations is $(x_1, 3, z_1)$.

$$2x - y + 4z = {}^-2$$
$$x + 2y + 3z = 8$$
$${}^-2x + 3y - z = n$$

Find the value of x_1, z_1 and n.

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2a) Explain why (2, 0, 3) is a solution to the set of simultaneous equations for all values of k.

$$x + ky + 8z = 26$$
$$3x + 5y - 4z = {}^-6$$
$$x + 2y + z = 5$$

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b) i) Find the value of k, such that there is a solution in which $y = 7$.

ii) Is this a unique solution to the system? Explain.

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